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The Cauchy problem for the wave equation with Lévy noise initial data

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Abstract

In this paper we study the Cauchy problem for the wave equation with space-time Lévy noise initial data in the Kondratiev space of stochastic distributions. We prove that this problem has a strong and unique C^2-solution, which takes an explicit form. Our approach is based on the use of the Hermite transform.

Keywords and phrases: Lévy processes, white noise analysis, stochastic partial differential equation.

AMS 1991 classification: 60G51; 60H40; 60H15.

1 Introduction

The purpose of this paper is to solve stochastic wave equations of the form

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2}(t,x) - \Delta U(t,x) &= F(t,x), \quad t>0, \quad x \in \mathbb{R}^{\mathrm{n}} \\ U(0,x) &= G(x), \quad x \in \mathbb{R}^{\mathrm{n}} \\ \frac{\partial U}{\partial t}(0,x) &= H(x), \quad x \in \mathbb{R}^{\mathrm{n}}. \end{aligned} \tag{1.1}$$

Here $\Delta U(t,x) = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2}(t,x)$ is the Laplacian with respect to $x = (x_1, \ldots, x_n) \in \mathbb{R}^{\mathrm{n}}$ and $F(t,x)$, $G(x)$ and $H(x)$ are given stochastic distribution valued (i.e. $(\mathcal{S})_{-1}$-valued) functions. The stochastic distribution space $(\mathcal{S})_{-1}$ is the *Lévy white noise analogue* of the standard Kondratiev spaces $(\mathcal{S})_{-1}$ (see Section 2.2 for definitions). In particular, equation (1.1) contains the special case where $G(x) = H(x) = 0$ and

$$F(t,x) = \overset{\bullet}{\eta}(t,x) = \frac{\partial^{n+1}\eta}{\partial t \partial x_1 \ldots \partial x_n}(t,x)$$

and is the *space-time Lévy white noise* ($\eta(t,x)$ is the space-time Lévy process/field).

We show that (1.1) has a unique $(\mathcal{S})_{-1}$-valued solution $U(t,x)$ (under certain smoothness conditions on F, G and H). See Theorems 3.7, 3.15 and 3.18.

Stochastic partial differential equations driven by *classical Brownian* space-time white noise were first studied by Walsh [W]. He considered a different solution concept than ours: A solution $U(t,x) = U(t,x,\omega)$ in the sense of Walsh is a *classical* distribution with respect to t and x for a.a. ω, and it satisfies the equation in classical distribution sense, for a.a. ω.

Our solution $U(t,x) = U(t,x,\omega)$ on the other hand, is a *stochastic* distribution in ω, for each t and x, and it satisfies the equation in the strong sense with respect to t and x, as a stochastic distribution valued (i.e. $(\mathcal{S})_{-1}$-valued) function.

With this last solution concept in mind, the stochastic wave equation driven by the classical Brownian white noise was solved for $n = 1$ and $n = 3$ by Jacobsen [Ja]. Our paper may be regarded as a Lévy white noise analogue of [Ja], extended to all $n = 1, 2, 3, \ldots$.

In order to achieve the corresponding existence and uniqueness results, we need a multi-parameter Lévy white noise calculus, including the method of Hermite transform. This is given in Section 2 and 3. We believe that this general machinery is useful for a large class of stochastic partial differential equations driven by Lévy space-time noise, and it is therefore of independent interest. Finally, in Section 3 we state and prove our existence and uniqueness results for equation (1.1).

2 Framework

In this section we recall some definitions and results in [LØP], which will be used later on to solve the Cauchy problem for the wave equation driven by Lévy white noise. We adopt the presentation and notation in [HØUZ], where the authors deal with Gaussian white noise theory. As basic references to white noise theory we recommend the worth reading books [HKPS], [Ku] and [O].

2.1 d-parameter Lévy process, chaos expansion

In this paper we are primarely interested in (d-parameter) pure jump Lévy processes without drift.

A Lévy process $\eta(t)$ on $\mathbb{R}_+$ is defined to be a stochastic process with independent and stationary increments starting at zero, i.e. $\eta(0) = 0$. Such a process possesses a càdlàg version (see [B]). The general structure of a Lévy process $\eta(t)$ is described by the celebrated Lévy-Khintchine formula, that is, $\eta(t)$ is uniquely determined in distribution by its characteristic function

$$E \exp(i\lambda\eta(t)) = \exp(-t\Psi(\lambda)); \lambda \in \mathbb{R} \tag{2.1}$$

with characteristic exponent

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda z} + i\lambda z\chi_{\{|z|<1\}})\nu(dz), \tag{2.2}$$

for constants $a \in \mathbb{R}$ and $\sigma \geq 0$. The measure ν is called *Lévy measure*, which gives information about the size and kind of the jumps of $\eta(t)$. The reader, who wants to know more about Lévy processes, is referred to [B] and [Sa].

From now on we solely consider pure jump Lévy processes without drift. Such processes can be looked upon as elements of the Poisson space $\widetilde{\mathcal{S}}(X)$ (see

[LØP]). We briefly explain the construction of the space $\widetilde{\mathcal{S}}(X)$. For details we refer to [LØP].

As is common we indicate by $\mathcal{S}'(\mathbb{R}^d), d \in \mathbb{N}$, the space of tempered distributions. $\mathcal{S}'(\mathbb{R}^d)$ is the dual of the space of rapidly decreasing functions or Schwartz space $\mathcal{S}(\mathbb{R}^d)$ (see for definitions [GV]). Let us choose the Hermite functions, denoted by $\{\xi_n\}_{n\geq 0}$, as a complete orthonormal system of $L^2(\mathbb{R})$. The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be topologized by the following compatible system of norms

$$\|\varphi\|_\gamma^2 := \sum_{\alpha\in\mathbb{N}^d} (1+\alpha)^{2\gamma}(\varphi,\xi_\alpha)^2_{L^2(\mathbb{R}^d)},\ \gamma\in\mathbb{N}_0^d,$$

where $\xi_\alpha := \prod_{i=1}^d \xi_{\alpha_i}$ and $(1+\alpha)^{2\gamma} := \prod_{i=1}^d (1+\alpha_i)^{2\gamma_i}$ for $\alpha = (\alpha_1,...,\alpha_d)\in\mathbb{N}^d$ and $\gamma = (\gamma_1,...,\gamma_d)\in\mathbb{N}_0^d$. Further, denote by $\|\varphi\|_{\gamma_i}$ a numbering of the norms in (2.1.2). Then we obtain a sequence of non-decreasing pre-Hilbertian norms $\|\varphi\|_p$, $p\in\mathbb{N}$, on the Schwartz space, by defining $\|\varphi\|_p = \sum_{i=1}^p \|\varphi\|_{\gamma_i}$. These norms are equivalent to the norms

$$\|\varphi\|_{q,\infty} := \sup_{0\leq k,|\gamma|\leq q}\ \sup_{z\in\mathbb{R}^d}\left|(1+|z|^k)\partial^\gamma\varphi(z)\right|,\ q\in\mathbb{N}_0,$$

where $\partial^\gamma\varphi = \frac{\partial^{|\gamma|}}{\partial z_1^{\gamma_1}...\partial z_d^{\gamma_d}}\varphi$ for $\gamma=(\gamma_1,...,\gamma_d)\in\mathbb{N}_0^d$ with $|\gamma| := \gamma_1+...+\gamma_d$.

In the sequel, let $X = \mathbb{R}^d\times\mathbb{R}_0$, where $\mathbb{R}_0 := \mathbb{R}-\{0\}$. We define the space $\mathcal{S}(X)$ by

$$\mathcal{S}(X) := \left\{\varphi\in\mathcal{S}(\mathbb{R}^{d+1}) : \varphi(z_1,...,z_d,0) = (\frac{\partial}{\partial z_{d+1}}\varphi)(z_1,...,z_d,0) = 0\right\}.$$

$\mathcal{S}(X)$ is a (countably Hilbertian) nuclear space with respect to the restriction of the norms $\|\cdot\|_p$, since it is a closed subspace of $\mathcal{S}(\mathbb{R}^{d+1})$. It turns out that $\mathcal{S}(X)$ is even a nuclear algebra, that is, $\mathcal{S}(X)$ is in addition a topological algebra with respect to the multiplication of functions. In the following, $\lambda^{\times d}$ will stand for the Lebesgue measure on $\mathbb{R}^d$ and ν for a Lévy measure on $\mathbb{R}_0$. We set $\pi = \lambda^{\times d}\times\nu$. We shall note that one could replace ν by a Radon measure on a topological space to develop a more general theory. This can be done without significant changes in our approach. It can be easily shown that there exists an element $1\otimes\dot{\nu}$ in $\mathcal{S}'(X)$ such that

$$\left\langle 1\otimes\dot{\nu},\phi\right\rangle = \int_X \phi(y)\pi(dy)$$

for all $\phi\in\mathcal{S}(X)$, where $\left\langle 1\otimes\dot{\nu},\phi\right\rangle = (1\otimes\dot{\nu})(\phi)$ is the action of $1\otimes\dot{\nu}$ on ϕ. We use the suggestive notation $\dot{\nu}$ to indicate that $\dot{\nu}$ is the Radon-Nikodym derivative of ν in a generalized sense. Further, we define the closed ideal $\mathcal{N}_\pi$ of $\mathcal{S}(X)$ by

$$\mathcal{N}_\pi := \{\phi\in\mathcal{S}(X) : \|\phi\|_{L^2(\pi)} = 0\}.$$

Finally, the space $\widetilde{\mathcal{S}}(X)$ is defined as the quotient ring

$$\widetilde{\mathcal{S}}(X) = \mathcal{S}(X)/\mathcal{N}_\pi.$$

The space $\widetilde{\mathcal{S}}(X)$ forms a (countably Hilbertian) nuclear algebra with the following compatible system of norms

$$\left\|\widehat{\phi}\right\|_{p,\pi} := \inf_{\psi\in\mathcal{N}_\pi}\|\phi+\psi\|_p,\quad p\in\mathbb{N},$$

see [LØP].

Denote by $\widetilde{\mathcal{S}}'(X)$ the dual of $\widetilde{\mathcal{S}}(X)$. The Bochner-Minlos theorem ensures the existence of a probability measure μ on the Borel sets of $\widetilde{\mathcal{S}}'(X)$ such that its characteristic functional is Poissonian with intensity π, i.e. for all $\phi \in \widetilde{\mathcal{S}}(X)$ we have that

$$\int_{\widetilde{\mathcal{S}}'(X)} e^{i\langle \omega,\phi\rangle} d\mu(\omega) = \exp\left(\int_X (e^{i\phi} - 1) d\pi\right), \tag{2.3}$$

where $\langle \omega, \phi\rangle = \omega(\phi)$ is the action of $\omega \in \widetilde{\mathcal{S}}'(X)$ on $\phi \in \widetilde{\mathcal{S}}(X)$. We call the probability measure μ on $\Omega = \widetilde{\mathcal{S}}'(X)$ a *Lévy white noise probability measure.* We shall only mention here that μ satisfies the *first condition of analyticity* and that it is *non-degenerate* (see Lemma 2.1.5 and Remark 2.1.6 in [LØP]). The first property is essential for the existence of certain symmetric polynomials $C_n(\omega)$, called *generalized Charlier polynomials* (see [KDS]): Let $\alpha(x) = \log(1 + x)$ and assume $\phi \in \widetilde{\mathcal{S}}(X)$ satisfies $\phi(x) > -1$ (modulo $\mathcal{N}_\pi$). The function α is holomorphic at zero and invertible. Further, set $\widetilde{e}(\phi, \omega) = \frac{\exp\langle \omega,\alpha(\phi)\rangle}{E_\mu[e^{\langle \omega,\alpha(\phi)\rangle}]}$. Then the exponential $\widetilde{e}(\phi, \omega)$ can be expanded into a power series at zero in terms of generalized Charlier polynomials $C_n(\omega)$, i.e.

$$\widetilde{e}(\phi, \omega) = \sum_{n\geq 0} \frac{1}{n!} \left\langle C_n(\omega), \phi^{\otimes n}\right\rangle, \tag{2.4}$$

for all ϕ in an open neighbourhood of zero in $\widetilde{\mathcal{S}}(X)$, where $\phi^{\otimes n} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$ (n-th symmetric tensor product of $\widetilde{\mathcal{S}}(X)$ with itself). The elements of this space can be interpreted as functions $f \in \mathcal{S}(X^n)$ modulo $\mathcal{N}_{\pi^{\times n}}$ such that $f = f(x_1, ..., x_n)$ is symmetric with respect to the variables $x_1, ..., x_n \in X$. The system $\{\langle C_n(\omega), \phi^{(n)}\rangle : \phi^{(n)} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}, n \in \mathbb{N}_0\}$ forms a total set in $L^2(\mu)$ and for all n, m, $\phi^{(n)} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} m}$ the orthogonality relation

$$\int\limits_{\widetilde{\mathcal{S}}'(X)} \left\langle C_n(\omega), \phi^{(n)}\right\rangle \left\langle C_m(\omega), \psi^{(m)}\right\rangle d\mu(\omega) = \delta_{n,m} n! (\phi^{(n)}, \psi^{(n)})_{L^2(X^n)} \tag{2.5}$$

is fulfilled. Now, for functions $f : X^n \longrightarrow \mathbb{R}$, define the *symmetrization* $(f)^\wedge$ of f by

$$(f)^\wedge(x_1, ..., x_n) := \frac{1}{n!} \sum_\sigma f(x_{\sigma_1}, ..., x_{\sigma_n}),$$

where the sum runs over all permutations σ on $\{1, ..., n\}$. Then a function $f : X^n \longrightarrow \mathbb{R}$ is symmetric, if and only if $\widehat{f} = f$. Define $\widehat{L}^2(X^n, \pi^{\times n})$ as the space of all symmetric functions on X^n, being square integrable with respect to $\pi^{\times n}$. The orthogonality relation (2.5) and the density of $\mathcal{S}(X)$ in $L^2(X, \pi)$ (see [LP]) enables us to extend the functional $\langle C_n(\omega), f_n \rangle$ from $f_n \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$ to $f_n \in \widehat{L}^2(X^n, \pi^{\times n})$. Further, we can indentify the polynomial $C_1(\omega)$ with $\omega - 1 \otimes \dot{\nu}$ (see [LP]). Thus we obtain by (2.5) the isometry

$$\int_{\widetilde{\mathcal{S}}'(X)} \langle \omega - 1 \otimes \dot{\nu}, f\rangle^2 \, d\mu(\omega) = \|f\|^2_{L^2(\pi)} \tag{2.6}$$

for all $f \in \widetilde{\mathcal{S}}(X)$. Next define for Borelian $\Lambda_1 \subset \mathbb{R}^d$, $\Lambda_2 \subset \mathbb{R}_0$ with $\pi(\Lambda_1 \times \Lambda_2) < \infty$ the random measures

$$N(\Lambda_1, \Lambda_2) := \langle \omega, \chi_{\Lambda_1 \times \Lambda_2} \rangle \text{ and } \widetilde{N}(\Lambda_1, \Lambda_2) := \langle \omega - 1 \otimes \dot{\nu}, \chi_{\Lambda_1 \times \Lambda_2} \rangle \,.$$

Their characteristic functions show that N is a Poisson random measure and $\widetilde{N}$ is the corresponding compensated Poisson random measure, where π is the compensator of $N(\Lambda_1, \Lambda_2)$. So we can naturally define the stochastic integral of $\phi \in L^2(\pi)$ with respect to $\widetilde{N}$ by

$$\int_X \phi(x,z)\widetilde{N}(dx,dz) := \langle \omega - 1 \otimes \dot{\nu}, \phi \rangle \,. \tag{2.7}$$

Based on (2.7) we finally define the *$d-$parameter Lévy process* or *space-time Lévy process,* denoted by $\eta(x)$, to be a càdlàg version of the random field

$$\widetilde{\eta}(x) := \int_X \chi_{[0,x_1]\times\ldots\times[0,x_d]}(y) \cdot z\widetilde{N}(dy,dz) \text{ for } x = (x_1, ..., x_d) \in \mathbb{R}^d,$$

where $[0, x_i]$ is interpreted as $[x_i, 0]$, if $x_i < 0$ and where it is assumed that the second moment with respect to the Lévy measure ν exists.

In conclusion we state a chaos expansion result in terms of generalized Charlier polynomials (see Theorem 2.2.1 in [LØP]). For this purpose we have to introduce some notation.

In the following we denote by $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$ the collection of all sequences $\alpha = (\alpha_1, \alpha_2, ...)$ with finitely many non-zero elements $\alpha_i \in \mathbb{N}_0$. Next define $Index(\alpha) = \max\{i : \alpha_i \neq 0\}$ and $|\alpha| = \sum_i \alpha_i$ for $\alpha \in \mathcal{J}$.

Now, we need two families of orthogonal polynomials. First, let $\{\xi_k\}_{k\geq 1}$ be the Hermite functions as before. Further, take a bijection $h : \mathbb{N}^d \longrightarrow \mathbb{N}$. Then we define the function $\zeta_k(x_1, ..., x_d) = \xi_{i_1}(x_1) \cdot ... \cdot \xi_{i_d}(x_d)$, if $k = h(i_1, ..., i_d)$ for $i_j \in \mathbb{N}$. Thus $\{\zeta_k\}_{k\geq 1}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$.

We intend to construct the second family of orthogonal polynomials. For this reason we impose the following integrability condition on the Lévy measure (see [NS]): For every $\varepsilon > 0$ there exists a $\lambda > 0$ such that

$$\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \exp(\lambda\,|z|)\nu(dz) < \infty. \tag{2.8}$$

This condition entails the existence of all moments ≥ 2 with respect to the Lévy measure ν. Let $\{l_m\}_{m\geq 0}$ be the orthogonalization of $\{1, z, z^2, ...\}$ with respect to the innerproduct of $L^2(\varrho)$, where $\varrho(dz) = z^2\nu(dz)$. Then the polynomials

$$p_m(z) := \frac{1}{\|l_{m-1}\|_{L^2(\varrho)}} z \cdot l_{m-1}(z)$$

constitute a complete orthogonal system in $L^2(\nu)$ (see [ØP]). In view of the following we shall stress that we could also have chosen any orthogonal basis in $\mathcal{S}(X) \subset L^2(\nu)$ for $d = 0$ to represent $p_m(z)$. This choice would cancel condition (2.8). However we use $p_m(z)$ to simplify the notation. Now define the bijective map

$$\kappa : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N};\ (i,j) \longmapsto j + (i + j - 2)(i + j - 1)/2. \tag{2.9}$$

Then, if $k=\kappa(i,j)$ for $i,j\in\mathbb{N}$, let

$$\delta_k(x,z)=\zeta_i(x)p_j(z).$$

Further, we set $Index(\alpha)=j$ and $|\alpha|=m$ for $\alpha\in\mathcal{J}$ and introduce the function $\delta^{\otimes\alpha}$ given by

$$\delta^{\otimes\alpha}((x_1,z_1),\dots,(x_m,z_m))=$$

$$\delta_1^{\otimes\alpha_1}\otimes\dots\otimes\delta_j^{\otimes\alpha_j}((x_1,z_1),\dots,(x_m,z_m))=\delta_1(x_1,z_1)\cdot\ldots\cdot\delta_1(x_{\alpha_1},z_{\alpha_1})$$

$$\cdot\ldots\cdot\delta_j(x_{\alpha_1+\ldots+\alpha_{j-1}+1},z_{\alpha_1+\ldots+\alpha_{j-1}+1})\cdot\ldots\cdot\delta_j(x_m,z_m),$$

where the terms with zero-components α_i are set equal to 1 in the product ($\delta_i^{\otimes 0}=1$). Then we define the *symmetrized tensor product* of the δ_k's, denoted by $\delta^{\widehat{\otimes}\alpha}$, as

$$\begin{aligned}\delta^{\widehat{\otimes}\alpha}((x_1,z_1),\dots,(x_m,z_m)) &= (\delta^{\otimes\alpha})^\wedge((x_1,z_1),...,(x_m,z_m))\\ &= \delta_1^{\widehat{\otimes}\alpha_1}\widehat{\otimes}\dots\widehat{\otimes}\delta_j^{\widehat{\otimes}\alpha_j}((x_1,z_1),...,(x_m,z_m)).\end{aligned}$$

Finally, we define for $\alpha\in\mathcal{J}$

$$K_\alpha(\omega):=\left\langle C_{|\alpha|}(\omega),\delta^{\widehat{\otimes}\alpha}\right\rangle,$$

where $K_0(\omega):=1$.

With the above notation we are ready to state the following chaos expansion result (see [LØP]).

Theorem 2.1. *The family $\{K_\alpha\}_{\alpha\in\mathcal{J}}$ forms an orthogonal basis in $L^2(\mu)$ with norm expression*

$$\|K_\alpha\|^2_{L^2(\mu)}=\alpha!:=\alpha_1!\alpha_2!...,$$

for $\alpha=(\alpha_1,\alpha_2,...)\in\mathcal{J}$. Thus, every $F\in L^2(\mu)$ can be uniquely represented as

$$F=\sum_{\alpha\in\mathcal{J}}c_\alpha K_\alpha$$

where $c_\alpha\in\mathbb{R}$ for all α and where we set $c_0=E[F]$.

Moreover, the following isometry holds:

$$\|F\|^2_{L^2(\mu)}=\sum_{\alpha\in\mathcal{J}}\alpha!c_\alpha^2.$$

2.2 Kondratiev spaces, Levy white noise, Hermite transform

First we recall the construction of some stochastic test function spaces and distribution spaces (see [LØP]), which are Lévy versions of the Kondratiev spaces, originally studied in [K]. More information about these spaces in the Gaussian setting can be found in [AKS] and [KLS].

Choose $0 \leq \rho \leq 1$, $k \in \mathbb{N}_0$ and define for an expansion $f = \sum_{\alpha \in \mathcal{J}} c_\alpha K_\alpha \in L^2(\mu)$ the norm

$$\|f\|^2_{\rho,k} := \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1+\rho} c_\alpha^2 (2\mathbb{N})^{k\alpha},$$

where $(2\mathbb{N})^{k\alpha} = (2 \cdot 1)^{k\alpha_1}(2 \cdot 2)^{k\alpha_2}...(2 \cdot m)^{k\alpha_m}$, if $Index(\alpha) = m$.

Further, set $(\mathcal{S})_{\rho,k} = \{f : \|f\|_{\rho,k} < \infty\}$ Then we define the *test function space* $(\mathcal{S})_\rho$ by

$$(\mathcal{S})_\rho = \bigcap_{k \in \mathbb{N}_0} (\mathcal{S})_{\rho,k},$$

We topologize this space by the projective topology.

Analogously, define for a formal expansion $F = \sum_{\alpha \in \mathcal{J}} b_\alpha K_\alpha$ the norms

$$\|F\|^2_{-\rho,-k} := \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} c_\alpha^2 (2\mathbb{N})^{-k\alpha},\ k \in \mathbb{N}_0.$$

Let $(\mathcal{S})_{-\rho,-k} = \{F : \|F\|_{-\rho,-k} < \infty\}$ and define the *stochastic distribution space* $(\mathcal{S})_{-\rho}$ by

$$(\mathcal{S})_{-\rho} = \bigcup_{k \in \mathbb{N}_0} (\mathcal{S})_{-\rho,-k},$$

endowed with the inductive topology. The space $(\mathcal{S})_{-\rho}$ is the dual of $(\mathcal{S})_\rho$ in virtue of the action

$$\langle F, f \rangle = \sum_{\alpha \in \mathcal{J}} b_\alpha c_\alpha \alpha!$$

for $F = \sum_{\alpha \in \mathcal{J}} b_\alpha K_\alpha \in (\mathcal{S})_{-\rho}$ and $f = \sum_{\alpha \in \mathcal{J}} b_\alpha K_\alpha \in (\mathcal{S})_\rho$. For general $0 \leq \rho \leq 1$ we have the following chain of spaces

$$(\mathcal{S})_1 \subset (\mathcal{S})_\rho \subset (\mathcal{S})_0 \subset L^2(\mu) \subset (\mathcal{S})_{-0} \subset (\mathcal{S})_{-\rho} \subset (\mathcal{S})_{-1}$$

The space $(\mathcal{S}) := (\mathcal{S})_0$ resp.$(\mathcal{S})^* := (\mathcal{S})_{-0}$ is a Lévy version of the *Hida test function space* resp. *Hida stochastic distribution space.* See [HKPS] and [HØUZ] for related spaces in Gaussian and Poissonian analysis.

One of the fundamental objects in Gaussian white noise analysis is the Gaussian white noise, which can be regarded as the time derivative of Brownian motion. Similarly to the Gaussian case we can construct the *Lévy white noise* on the Hida distribution space $(\mathcal{S})^*$ (see Definition 2.2.4 in [LØP]). We define the *(d-parameter) Lévy white noise* $\overset{\bullet}{\eta}(x)$ of the Lévy process $\eta(x)$ by the formal expansion

$$\overset{\bullet}{\eta}(x) = m \sum_{k \geq 1} \zeta_k(x) K_{\epsilon^{\kappa(k,1)}}$$

where $\zeta_k(x)$ is defined by Hermite functions, $\kappa(i,j)$ is the map in (2.9), $m := \|z\|_{L^2(\nu)}$ and where $\epsilon^l \in \mathcal{J}$ is defined by

$$\epsilon^l(j) = \begin{cases} 1 \text{ for } j = l \\ 0 \text{ else} \end{cases}, l \geq 1$$

The uniform boundedness of the Hermite functions (see e.g. [T]) implies that the Lévy white noise $\overset{\bullet}{\eta}(x)$ takes values in $(\mathcal{S})^*$ for all x. Since the d-parameter Lévy process $\eta(x)$ can be written as

$$\eta(x) = \sum_{k\geq 1} m \int_0^{x_d} \ldots \int_0^{x_1} \zeta_k(y_1, \ldots y_d)\, dy_1 \ldots dy_d \cdot K_{\epsilon^{\kappa(k,1)}},$$

we can interprete $\overset{\bullet}{\eta}(x)$ as the time-space derivative of $\eta(x)$ in $(\mathcal{S})^*$, i.e.

$$\frac{\partial^d}{\partial x_1 \ldots \partial x_d} \eta(x) = \overset{\bullet}{\eta}(x) \text{ in } (\mathcal{S})^*.$$

Thus we are entitled to call $\overset{\bullet}{\eta}(x)$ white noise. Let us note that one can naturally generalize the concept of Lévy white noise, by defining the *($d-$parameter) white noise* $\overset{\bullet}{\widetilde{N}}(x, z)$ of the Poisson random measure $\widetilde{N}(dx, dz)$ (see [ØP]). Then $\overset{\bullet}{\eta}(x)$ can be expressed by $\overset{\bullet}{\widetilde{N}}(x, z)$ as

$$\overset{\bullet}{\eta}(x) = \int_{\mathbb{R}} z \overset{\bullet}{\widetilde{N}}(x, z) \nu(dz),$$

where the right side is given in terms of a Bochner integral with respect to ν.

Next we introduce a *(stochastic) Wick product* on the space $(\mathcal{S})_{-1}$ with respect to the white noise measure μ (see [LØP]). For more general information about the (Gaussian or Poissonian) Wick product the reader may consult e.g. [HKPS], [DM].

The *Lévy Wick product*, denoted by the symbol $\diamond$, is defined by

$$(K_\alpha \diamond K_\beta)(\omega) = (K_{\alpha+\beta})(\omega), \ \alpha, \beta \in \mathcal{J}$$

and extended linearly (see Definition 2.3.1 in [LØP]). Then, e.g., if $f_n \in \widehat{L}^2(\pi^{\times n})$ and $g_m \in \widehat{L}^2(\pi^{\times m})$ we have

$$\langle C_n(\omega), f_n \rangle \diamond \langle C_m(\omega), g_m \rangle = \left\langle C_{n+m}(\omega), f_n \widehat{\otimes} g_m \right\rangle.$$

Note that the spaces $(\mathcal{S})_1$, $(\mathcal{S})_{-1}$ and $(\mathcal{S}), (\mathcal{S})^*$ are topological algebras with respect to the Lévy Wick product $\diamond$ (see [LØP]). An important feature of the Wick product is that it can be related to Itô-Skorohod integrals. More precisely, this relation can be expressed as

$$\int_0^T Y(t) \delta\eta(t) = \int_0^T Y(t) \diamond \overset{\bullet}{\eta}(t) dt, \tag{2.10}$$

if $Y(t) = Y(t, \omega)$ is Skorohod integrable (see [DØP]). The left side is the Skorohod integral of $Y(t)$, whereas the integral on the right is the Bochner-integral on $(\mathcal{S})^*$. The Skorohod integral extends the Itô integral in the sense that both integrals coincide, if $Y(t, \omega)$ is adapted.

The *Hermite transform* was first introduced by Lindstrøm et al. (1991) [LØU] in the Gaussian and Poissonian case and it has proved to be a useful tool in the study of stochastic (partial) differential equations (see e.g. [HØUZ]). Its definition in the Lévy case is analogous. Let $F = \sum_{\alpha\in\mathcal{J}} a_\alpha K_\alpha \in (\mathcal{S})_{-1}$ with $a_\alpha \in \mathbb{R}$. The *Lévy Hermite transform of* F, denoted by $\mathcal{H}F$, is defined by

$$\mathcal{H}F(z) = \sum_{\alpha\in\mathcal{J}} a_\alpha z^\alpha \in \mathbb{C}, \tag{2.11}$$

if convergent, where $z = (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequences of complex numbers) and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n} ...,$$

for $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{J}$ with the convention $z_j^0 = 1$. Since

$$\overset{\bullet}{\eta}(x) = m \sum_{k\geq 1} \zeta_k(x) K_{\epsilon^{\kappa(k,1)}},$$

the Hermite transform of the d-parameter Lévy white noise can be calculated as

$$\mathcal{H}(\overset{\bullet}{\eta})(x, z) = m \sum_{k\geq 1} \zeta_k(x) \cdot z_{\kappa(k,1)},$$

which is convergent for all $z \in \left(\mathbb{C}^{\mathbb{N}}\right)_c$ (the set of all finite sequences in $\mathbb{C}^{\mathbb{N}}$). The Hermite transform is an algebra homomorphism between $(\mathcal{S})_{-1}$ and the algebra of power series in infinitely many complex variables. In particular, it converts the Wick product into ordinary products, that is

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}(F)(z) \cdot \mathcal{H}(G)(z)$$

for F, $G \in (\mathcal{S})_{-1}$ and all z such that $\mathcal{H}(F)(z)$ and $\mathcal{H}(G)(z)$ exist. Next, let us define for $0 < R, q < \infty$ the infinite-dimensional neighborhoods $K_q(R)$ in $\mathbb{C}^{\mathbb{N}}$ by

$$K_q(R) = \{(\xi_1, \xi_2, ...) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\neq 0} |\xi^\alpha|^2 (2\mathbb{N})^{q\alpha} < R^2\}.$$

We conclude this section with a characterization theorem for the space $(\mathcal{S})_{-1}$ (compare Theorem 2.6.11 in [HØUZ]).

Theorem 2.2. *The following statements hold*

(i) *Let* $F = \sum_{\alpha\in\mathcal{J}} a_\alpha K_\alpha \in (\mathcal{S})_{-1}$, *then there exist* q, $M_q < \infty$ *such that*

$$|\mathcal{H}F(z)| \leq \sum_{\alpha\in\mathcal{J}} |a_\alpha| |z^\alpha| \leq M_q (\sum_{\alpha\in\mathcal{J}} (2\mathbb{N})^{q\alpha} |z^\alpha|^2)^{\frac{1}{2}}$$

for all $z \in \left(\mathbb{C}^{\mathbb{N}}\right)_c$. *In particular,* $\mathcal{H}F$ *is a bounded analytic function on* $K_q(R)$ *for all* $R < \infty$.

(ii) *Conversely, suppose that* $g(z) = \sum_{\alpha\in\mathcal{J}} b_\alpha z^\alpha$ *is a power series of* $z \in \left(\mathbb{C}^{\mathbb{N}}\right)_c$ *such that there exist* $q < \infty$, $\delta > 0$ *with* $g(z)$ *absolutely convergent and bounded on* $K_q(\delta)$. *Then there exists a unique* $G \in (\mathcal{S})_{-1}$ *such that* $\mathcal{H}G = g$, *namely*

$$G = \sum_{\alpha\in\mathcal{J}} b_\alpha K_\alpha$$

3 The Cauchy problem for the wave equation driven by Lévy space-time white noise

Partial differential equations perturbed in some way by space-time white noise appear in many physical and engineering problems. For example the problem of stabilization of systems in automatic control theory has been investigated for Gaussian white noise with respect to elliptic, parabolic and hyperbolic partial differential equations (see [A], [AW]). Other areas are neurophysiology, interest rate modeling in finance or the study of amorphous thin-film growth (see e.g. [W], [S] and [BH]). Recently there has been an increased interest in the more general Lévy noise, see e.g. [M], [AW] and the references therein. One way of which Lévy noise occurs is in the so-called Schrödinger problem of probabilistic evolution. While the non-relativistic theory leads to Wiener noise, several relativistic Hamiltonians are known to generate Lévy noise. We also note that the Cauchy problem in two dimensions for the wave equation with smooth ordinary functions as initial data and driven by a Lévy point process has been studied in [DH].

Our approach to solve system (1.1) can be outlined as follows: First we study the homogenous case, i.e. the forcing term $F(t,x) = 0$ in (1.1). We convert (1.1) into a *deterministic* system of partial differential equations with complex coefficients, by applying the Hermite transform (2.11). Then, if we are able to determine a solution of the resulting PDE, we will take the inverse Hermite transform of it to solve the original equation. Afterwards we consider the inhomogenous wave equation with initial values equal to zero, i.e. the forcing term F is a stochastic distribution process and $G(x) = F(x) = 0$ in (1.1). We solve this problem in the same manner as in the homogeneous case. Finally, it is verified that the sum of the solutions of the latter two problems supplies a solution for the general case.

In Section 3.1 we derive solutions for system (1.1) in the case of space dimension 1 and then, in section 3.2, we solve the problem in any dimension n.

In the following we say that an $(\mathcal{S})_{-1}$-process $F(x)$ is strongly integrable in $(\mathcal{S})_{-1}$ over a 1-dimensional interval I_1 if the associated Riemann sums converge in $(\mathcal{S})_{-1}$. The limit is written

$$\lim_{n\to\infty} \sum_{k=1}^{n} F(t_k^*)\Delta t_k^n \overset{(\mathcal{S})_{-1}}{=} \int_{I_1} F(t)\, dt.$$

For a rectangle, $I \subset \mathbb{R}^n$, the integral is defined repeatedly as

$$\int_I F(x)\, dx \;=\; \int_{I_n} \dots \int_{I_1} F(x)\, dx_1 \dots dx_n$$

We shall follow the common practice to indicate by $C^k(G, (\mathcal{S})_{-1})$ the space of continuous functions $f : G \mapsto (\mathcal{S})_{-1}$, which have continuous derivatives up to order k. Here, G is an open subset of $\mathbb{R}^n$.

Let $f : G \times K_q(\delta) \ni (x,z) \mapsto \mathbb{C}$. The following properties will frequently occur:

P1	f is bounded on every $K \times K_q(\delta)$, where $K \subset G$ is compact
P2	f is continuous in x for fixed z
P2'	f is continuous in x uniformly over $K_q(\delta)$
P3	f is analytic in z for fixed x

Note that P1 just means bounded in case G is compact.

3.1 1-dimensional wave equation

In this section we investigate the stochastic wave equation in one space dimension. For this reason we distinguish between the following two subcases of the problem to obtain the general solution.

3.1.1 Homogenous case

First we solve the initial value problem of the homogeneous wave equation, i.e. we aim finding a solution for

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) &= 0 \\ U(0,x) &= G(x),\ G(x) \in C^2(\mathbb{R}, (\mathcal{S})_{-1}) \\ \frac{\partial U}{\partial t}(0,x) &= H(x),\ H(x) \in C^1(\mathbb{R}, (\mathcal{S})_{-1}) \end{aligned} \tag{3.1}$$

If we apply the Hermite transform to system (3.1) we get

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) &= 0 \\ u(0,x) &= g(x) \\ \frac{\partial u}{\partial t}(0,x) &= h(x) \end{aligned} \tag{3.2}$$

where the functions u, g and h indicate the corresponding Hermite transformed distributions. The same proof as for Theorem 2.8.1 in [HØUZ] implies that there exist q and δ such that $g \in C^1(\mathbb{R}, \mathbb{C})$ and $h \in C^2(\mathbb{R}, \mathbb{C})$. By comparing the real and imaginary parts in system (3.2) we obtain (see e.g. [ES] and [J])

$$u(t,x) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(s)ds \tag{3.3}$$

Then the inverse Hermite transform of (3.3) suggests itself as a natural candidate for a solution of system (3.1). So we have to check the existence of such inverse. A sufficient condition for the existence is provided by the following result.

Theorem 3.1. *Assume X is a function from a bounded, open set $D \subset \mathbb{R}_+ \times \mathbb{R}^d$ to $(\mathcal{S})_{-1}$ such that $\mathcal{H}X$ solves the Hermite transformed system (3.2) for all $(t,x,z) \in D \times K_q(\delta)$ for some $q < \infty$, $\delta > 0$. Furthermore let us require that the partial derivatives $\frac{\partial}{\partial t}\mathcal{H}X(t,x,z)$, $\frac{\partial^2}{\partial t^2}\mathcal{H}X(t,x,z)$ and $\frac{\partial^2}{\partial x_j^2}\mathcal{H}X(t,x,z)$, $j = 1,...,d$ satisfy the properties P1, P2 and P3. Then X solves equation (3.1) in the strong sense in $(\mathcal{S})_{-1}$.*

Proof Apply repeatedly the same proof of Lemma 2.8.4 in [HØUZ] to the case involving higher order derivatives. □

Next, we denote by $C^k(G, (\mathcal{S})_{-1})$ with open $G \subset \mathbb{R}_+ \times \mathbb{R}^d$ the space of functions belonging to C^k. For the proof of the main result of this subsection we have to make use of the following Lemmas.

Lemma 3.2. *The following assertions are equivalent:*

(i) $F_n \to F$ *in* $(\mathcal{S})_{-1}$
(ii) *There exist* $q < \infty$, $\delta > 0$ *such that*

$$\sup_{z \in K_q(\delta)} |\mathcal{H}F_n(z) - \mathcal{H}F(z)| \longrightarrow 0$$

Proof Same as in Theorem 2.8.1 in [HØUZ]. □

Lemma 3.3. *Let* $F : G \longrightarrow (\mathcal{S})_{-1}$. *Then the following are equivalent:*

(i) F *is continuous*
(ii) *There exist* $q < \infty$, $\delta > 0$ *such that* $\mathcal{H}F$ *satisfies P1, P2' and P3.*

Proof Let $V_{q\delta} = \{f : K_q(\delta) \to \mathbb{C}, \sup_{K_q(\delta)} |f| < \infty\}$.

(i)$\Longrightarrow$(ii). Fix an x_0 in a compact set $K \subset G$. By Lemma 3.2 we have that $\exists q' \, \exists \delta' \, \forall \varepsilon > 0 \, \exists \gamma' > 0$

$$x \in B(x_0, \gamma\prime) \cap G \Longrightarrow \sup_{z \in K_{q\prime}(\delta\prime)} |\mathcal{H}F(x) - \mathcal{H}F(x_0)| < \varepsilon.$$

Hence P2' holds. By compactness there exist q and δ such that $\mathcal{H}F(x) \in V_{q\delta}$ for all $x \in K$. This means P1 holds. P3 follows from Theorem 2.2.

(ii)$\Longrightarrow$(i). Let $x_n \to x$ in G and $F_n = F(x_n)$, $F = F(x)$. By condition P2' and Lemma 3.2 continuity follows. □

From this argument it also follows that $F \in C^k(G, (\mathcal{S})_{-1})$ if and only if $\mathcal{H}F \in C^k(G, \mathbb{C})$ for fixed $z \in K_q(\delta)$.

Lemma 3.4. *Let* $\mathbb{R} \supset [a, b] \ni t \mapsto F(t) \in (\mathcal{S})_{-1}$ *and suppose there exist* $q < \infty$ *and* $\delta > 0$ *such that* $\mathcal{H}F$ *satisfies P1 and P2. Then* $F(t)$ *is strongly integrable and*

$$\mathcal{H} \int_a^b F(t) dt = \int_a^b \mathcal{H}F(t) dt.$$

Proof Identical to the proof of Lemma 2.8.5 in [HØUZ]. □

In the homogenous case we attain the following result.

Theorem 3.5. *The initial value problem can be uniquely solved in* $C^2(\mathbb{R}_+ \times \mathbb{R}, (\mathcal{S})_{-1})$. *Its solution is explicitly given by*

$$U(t, x) = \frac{1}{2}(G(x + t) + G(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} H(s) ds$$

Proof Since the classical boundary value problem possesses a unique solution (see e.g. [ES], [J]), the uniqueness of the solution is a direct consequence of the characterization theorem (Theorem 2.2).

The proof of the existence of a solution boils down to the verification of the assumptions of Theorem 3.1. Since $\int_{x-t}^{x+t} h(s)ds$ satisfies P1, P2 and P3 (by Lemma 3.3), $u(t,x)$ comes up to the same properties. Finally, let us representively check the conditions for $u_t(t,x)$. The other partial derivatives can be tackled analogously. Differentiation with respect to t on both sides of (3.3) gives

$$u_t(t,x) = \frac{1}{2}(g'(x+t) - g'(x-t)) + \frac{1}{2}(h(x+t) + h(x-t))$$

for all $z \in K_q(\delta)$ with approriately chosen q, δ. By assumption and Lemma 3.3 it follows that $h(x)$ and $g'(x)$ fulfill P1, P2 and P3 for $z \in K_{q'}(\delta')$ and some q', δ'. So $u_t(t,x)$ satisfies the requirements of Theorem 3.1, too. Then the existence of the solution follows.

The claimed smoothness of the solution can be easily seen with the help of Lemma 3.3. □

3.1.2 Inhomogeneous case

First we look for a solution of the following initial value problem for the inhomogeneous wave equation

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) &= F(t,x) \in C^2(\mathbb{R}_+ \times \mathbb{R}, (\mathcal{S})_{-1}) \\ U(0,x) &= 0 \\ \frac{\partial U}{\partial t}(0,x) &= 0 \end{aligned} \tag{3.4}$$

Using again the Hermite transform in (3.4) we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) &= f(t,x) \in C^2(\mathbb{R}_+ \times \mathbb{R}, (\mathcal{S})_{-1}) \\ u(0,x) &= 0 \\ \frac{\partial u}{\partial t}(0,x) &= 0, \end{aligned}$$

for all $z \in K_q(\delta)$, where $f(t,x) = \mathcal{H}F(t,x)$.

A solution of this initial value problem is given by

$$u(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)dyds. \tag{3.5}$$

See e.g. [ES], [J].

By the same arguments as in the proof of Theorem 3.5 we can deduce the following result for the particular initial value problem.

Theorem 3.6. *The initial value problem (3.4) admits a unique solution in* $C^2(\mathbb{R}_+ \times \mathbb{R}, (\mathcal{S})_{-1})$, *which has the explicit form*

$$U(t,x) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} F(s,y)dyds.$$

Finally we intend to solve the general initial value problem for the inhomogenous wave equation, that is, we study

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) &= F(t,x) \in C^2(\mathbb{R}_+ \times \mathbb{R}, (\mathcal{S})_{-1}) \\ U(0,x) &= G(x) \in C^2(\mathbb{R}, (\mathcal{S})_{-1}) \\ \frac{\partial U}{\partial t}(0,x) &= H(x) \in C^1(\mathbb{R}, (\mathcal{S})_{-1}) \end{aligned} \tag{3.6}$$

Let us denote by U_h resp. U_p the solution in Theorem 3.5 resp. Theorem 3.6. Then a short calculation shows that $U = U_h + U_p$ supplies a solution for (3.6). It is easily seen that this solution also holds uniquely. Thus we proved

Theorem 3.7. *There exists a unique solution of system (3.6) in* $C^2(\mathbb{R}_+ \times \mathbb{R}, (\mathcal{S})_{-1})$. *This solution takes the explicit form*

$$\begin{aligned} U(t,x) &= \frac{1}{2}(G(x+t) + G(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} H(s)ds \\ &\quad + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} F(s,y)dyds \end{aligned}$$

Example 3.8. In the last Theorem chose $G(x) = x\xi,\ \xi \in L^2(\mu),\ H = 0$ and let $F(t,x) = \phi(t,x) \diamond \overset{\bullet}{\eta}(t,x)$ for a not necessarily predictable process ϕ with $E\int_{\mathbb{R}^2} \phi^2(t,x)\, d(t,x) < \infty$, where $\overset{\bullet}{\eta}(t,x)$ is the 2-parameter Lévy white noise. Then relation (2.10) and Theorem (3.7) entail that

$$U(t,x) = x\xi + \frac{1}{2}\int_{[0,t)\times[x-(t-s),x+(t-s))} \phi(s,y)\, d\eta(s,y).$$

3.2 n-dimensional wave equation

As in the deterministic case, we treat the problem differently whether or not the dimension is odd or even.

Since the change of variables formula holds for Bochner integrals, surface integrals of continuous $(\mathcal{S})_{-1}$-valued processes can be defined similarly as for R-valued ones. For the n-dimensional unit ball $\partial B^n = \{x \in \mathbb{R}^{\mathrm{n}} : \|x\| \leq 1\}$ with boundary ∂B^n this means that

$$\int_{\partial B^n} F(x)\, dS(x) = \int_Q F \circ \varphi(\theta) m(\theta)\, d\theta,$$

where $\varphi : Q \to \partial B^n$ maps Descartian coordinates onto spherical ones and the Jacobian m is continuous with values in $[0,1]$. Unambiguously we can write

$$\begin{aligned}\int_{\partial B^n} F(x)\, dS(x) &= \int_{B^{n-1}} F(x, \sqrt{1-|x|^2}) \frac{dx}{\sqrt{1-|x|^2}} \\ &\quad + \int_{B^{n-1}} F(x, -\sqrt{1-|x|^2}) \frac{dx}{\sqrt{1-|x|^2}}.\end{aligned}$$

In particular, if F does not depend on x_n then

$$\int_{\partial B^n} F(x)\,dS(x) = 2\int_{B^{n-1}} F(x)\frac{dx}{\sqrt{1-|x|^2}}. \tag{3.7}$$

We start by proving some lemmas essential for our main results Theorem 3.15 and 3.18.

Lemma 3.9. *For a rectangle $Q \subset \mathbb{R}^n$, let F map Q into $(\mathcal{S})_{-1}$ and suppose there exist $q < \infty$ and $\delta > 0$ such that $\mathcal{H}F$ satisfies P1 and P2. Then F is strongly integrable over Q and*

$$\mathcal{H}\int_Q F(x)\,dx = \int_Q \mathcal{H}F(x)\,dx.$$

Proof Apply Lemma 3.4 repeatedly to

$$x_k \mapsto \int \ldots \int F(x)\,dx_1 \ldots dx_{k-1}$$

holding $(x_{k+1}, \ldots, x_n)$ fixed and start with $k = 1$, i.e., $x_1 \mapsto F(x)$ for fixed $(x_2, \ldots, x_n)$. □

Lemma 3.10. *Suppose $F \in C(\partial B^n, (\mathcal{S})_{-1})$. Then $\int_{\partial B^n} F(x)\,dS(x) \in (\mathcal{S})_{-1}$ and*

$$\mathcal{H}\int_{\partial B^n} F(x)\,dS(x) = \int_{\partial B^n} \mathcal{H}F(x)\,dS(x)$$

Proof Since $F \circ \varphi \cdot m$ is continuous on Q, Lemma 3.3 and 3.9 shows that $\int_Q F \circ \varphi(\theta) m(\theta)\,d\theta \in (\mathcal{S})_{-1}$ and

$$\mathcal{H}\int_Q F \circ \varphi(\theta) m(\theta)\,d\theta = \int_Q f \circ \varphi(\theta) m(\theta)\,d\theta. \qquad \square$$

Lemma 3.11. *Let $G \subset \mathbb{R}^n$ be open and $\Psi : G \times \partial B^n \longrightarrow \mathbb{R}^n$ and $w : G \times \partial B^n \mapsto \mathbb{R}$ both continuous. Suppose there exist $q' < \infty$, $r' > 0$ such that f satisfies P1, P2' and P3 on G. Then there exist q, δ such that*

$$x \longmapsto \int_{\partial B^n} f \circ \Psi(x,y) w(x,y)\,dS(y)$$

satisfies P1, P2' and P3.

Proof P1 and P2' are immediately inherited. Theorem 2.2 assures that $\mathcal{H}F = f$ for some F continuous by Lemma 3.3. Lemma 3.10, applied to the map $y \mapsto F \circ \Psi(x,y) w(x,y)$, shows that $\int_{\partial B^n} F \circ \Psi(x,y) w(x,y)\,dS(y) \in (\mathcal{S})_{-1}$ for every x and its Hermite transform $\int_{\partial B^n} f \circ \Psi(x,y) w(x,y)\,dS(y)$ is analytic in some $K_q(\delta)$. □

Lemma 3.12. *For $F \in C^1(\partial B^n \times \mathbb{R}, (\mathcal{S})_{-1})$,*

$$\frac{d}{dr}\int_{\partial B^n} F(x,r)\,dS(x) \;=\; \int_{\partial B^n} \frac{d}{dr}F(x,r)\,dS(x).$$

Proof By Theorem 2.2 we may as well consider the Hermite transformed equation. Using P1 and P2 it readily follows from standard theorems governing the interchange of differentiation and integration of complex valued functions. □

3.2.1 Odd-dimensional wave equation

In this section the Cauchy problem is considered in odd dimensions.

We first focus on the *homogenous* case, i.e. the wave equation

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2}(t,x) - \Delta U(t,x) &= 0 \\ U(0,x) &= G(x),\ G \in C^{(n+3)/2}(\mathbb{R}^n, (\mathcal{S})_{-1}) \\ \frac{\partial U}{\partial t}(0,x) &= H(x),\ H \in C^{(n+1)/2}(\mathbb{R}^n, (\mathcal{S})_{-1}) \end{aligned} \tag{3.8}$$

Uniqueness. Assume a solution U to (3.8) exists. Take the Hermite transform of the equation and suppress the dependence on z to get

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) &= 0 \\ u(0,x) &= g(x) \\ \frac{\partial u}{\partial t}(0,x) &= h(x) \end{aligned}$$

By considering real and imaginary parts separately this problem has the unique solution (see e.g. [Fo])

$$\begin{aligned} u(t,x) = C_n \Bigg[& \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)2} t^{n-2} \int_{\partial B^n} g(x+ty)\, dS(y) \\ & + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} t^{n-2} \int_{\partial B^n} h(x+ty)\, dS(y) \Bigg], \end{aligned}$$

where C_n is a positive constant only dependent on n. Note also that when $n = 3$ the differential operator is raised to the power 0 and should be interpreted as the identity operator. Moreover, u is a bounded analytical function on some $K_q(R)$ by Theorem 2.2 and the inverse Hermite transform U is unique.

Existence. Lemma 3.12 shows that it is possible to define

$$U(t,x) = C_n \Bigg[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} t^{n-2} \int_{\partial B^n} G(x+ty)\, dS(y) \tag{3.9}$$

$$+ \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} t^{n-2} \int_{\partial B^n} H(x+ty)\, dS(y) \Bigg]. \tag{3.10}$$

By Lemma 3.10 $\mathcal{H}U = u$. Let us check that the second partial derivatives of u are continuous in (t,x), analytic in z and bounded on $K_t \times K_x \times K_q(R)$ where K_t and K_x are compacts in $[0,\infty)$ and $\mathbb{R}^n$ respectively. The first order derivatives are similar/simpler to check. Interchanging differentiation and integration, a second derivative in x_k results in

$$\begin{aligned} \frac{\partial^2 u}{\partial x_k^2}(t,x) = C_n \Bigg[& \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} t^{n-2} \int_{\partial B^n} \frac{\partial^2}{\partial x_k^2} g(x+ty)\, dS(y) \\ & + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} t^{n-2} \int_{\partial B^n} \frac{\partial^2}{\partial x_k^2} h(x+ty)\, dS(y) \Bigg]. \end{aligned} \tag{3.11}$$

Carrying out the differentiations in t reveals that (3.11) is a linear combination of terms of type

$$t^m \int_{\partial B^n} y^\alpha \partial_k^2 \partial^\alpha f(x+ty)\, dS(y), f \in \{g,h\}$$

with $m \in \mathbb{N}$ and the multi index α, $|\alpha| \leq (n-1)/2$. Since $g \in C^{(n+3)/2}(\mathbb{R}^{\mathrm{n}})$, the integrand satisfies P1, P2' and P3 and similarly for the terms involving h, so, using Lemma 3.11 with $G = \mathbb{R} \times \mathbb{R}^{\mathrm{n}}$, $\Psi(t,x,y) = x + ty$ and $w(t,x,y) = t^m y^\alpha$, $\partial_k^2 u$ also has these properties. Since, finally, $\partial_t^2 u = \sum_k \partial_{x_k}^2 u$, Theorem 3.1 shows that U is the solution to the equation. Finally, as $t \to 0$ the solution and its time derivative tend to the initial values which follow by the same argument as in the deterministic case. Thus we have proved the following theorem.

Theorem 3.13. *The initial value problem (3.8) can be uniquely solved in* $C^2(\mathbb{R}_+ \times \mathbb{R}^{\mathrm{n}}, (\mathcal{S})_{-1})$. *Its solution is explicitly given by (3.9)*

We now turn to the *inhomogeneous* case and determine the solution of the initial value problem for the inhomogeneous wave equation

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) &= F(t,x) \in C^{(n+1)/2}(\mathbb{R}_+ \times \mathbb{R}^n, (\mathcal{S})_{-1}) \\ U(0,x) &= 0 \\ \frac{\partial U}{\partial t}(0,x) &= 0 \end{aligned} \tag{3.12}$$

The Hermite transform converts (3.12) to the system

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) &= f(t,x) \\ u(0,x) &= 0 \\ \frac{\partial u}{\partial t}(0,x) &= 0. \end{aligned}$$

A solution of this problem is given by the formula

$$u(t,x) \;=\; C_n \int_0^t \left[\left(\frac{1}{r} \frac{\partial}{\partial t} \right)^{(n-3)/2} r^{n-2} \int_{\partial B^n} f(t-r, x+ry)\, dS(y) \right] dr. \tag{3.13}$$

Using almost the same arguments as in the last section we can conclude

Theorem 3.14. *The initial value problem (3.12) can be uniquely solved in* $C^2(\mathbb{R}_+ \times \mathbb{R}^n, (\mathcal{S})_{-1})$, *by the process*

$$U(t,x) \;=\; C_n \int_0^t \left[\left(\frac{1}{r} \frac{\partial}{\partial t} \right)^{(n-3)/2} r^{n-2} \int_{\partial B^n} F(t-r, x+ry)\, dS(y) \right] dr.$$

Again, if we indicate by U_h and U_p the corresponding solutions in Theorem 3.13 and Theorem 3.14, one checks that $U = U_h + U_p$ gives a unique solution of the general initial value problem

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2}(t,x) - \Delta U(t,x) &= F(t,x) \in C^{(n+1)/2}(\mathbb{R}_+ \times \mathbb{R}^{\mathrm{n}}, (\mathcal{S})_{-1}) \\ U(0,x) &= G(x),\ G \in C^{(n+3)/2}(\mathbb{R}^{2n+1}, (\mathcal{S})_{-1}) \\ \frac{\partial U}{\partial t}(0,x) &= H(x),\ H \in C^{(n+1)/2}(\mathbb{R}^n, (\mathcal{S})_{-1}). \end{aligned} \tag{3.14}$$

So we obtain

Theorem 3.15. *System (3.14) allows a unique $C^2(\mathbb{R}_+ \times \mathbb{R}^n, (\mathcal{S})_{-1})$-solution of the form*

$$
\begin{aligned}
U(t,x) \;=\;& C_n \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t} \right)^{(n-3)/2} t^{n-2} \int_{\partial B^n} G(x+ty)\, dS(y) \right. \\
&+ \left(\frac{1}{t}\frac{\partial}{\partial t} \right)^{(n-3)/2} t^{n-2} \int_{\partial B^n} H(x+ty)\, dS(y) \\
&\left. + \int_0^t \left[\left(\frac{1}{r}\frac{\partial}{\partial t} \right)^{(n-3)/2} r^{n-2} \int_{\partial B^n} F(t-r, x+ry)\, dS(y) \right] dr \right\}.
\end{aligned}
$$

3.2.2 Even-dimensional wave equation

Theorem 3.16. *Assume n is even. If $G \in C^{(n+4)/2}(\mathbb{R}^n, (\mathcal{S})_{-1})$ and $H \in C^{(n+2)/2}(\mathbb{R}^n, (\mathcal{S})_{-1})$ then the solution to the homogeneous problem is*

$$
\begin{aligned}
U(t,x) \;=\;& C_n \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t} \right)^{(n-2)/2} t^{n-1} \int_{B^n} \frac{G(x+ty)}{\sqrt{1-|y|^2}}\, dy \right. \\
&\left. + \left(\frac{1}{t}\frac{\partial}{\partial t} \right)^{(n-2)/2} t^{n-1} \int_{B^n} \frac{H(x+ty)}{\sqrt{1-|y|^2}}\, dy \right\},
\end{aligned}
$$

where now $C_n = 2/[(n-1)!!\omega_{n+1}]$. Note that when $n = 2$ the differential operator is raised to the power 0 and is the identity operator.

Proof We use the method of descent. Considered in $\mathbb{R}^{n+1}$ the solution is

$$
\begin{aligned}
U(t,x) \;=\;& C_{n+1} \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t} \right)^{(n-2)/2} t^{n-1} \int_{\partial B^{n+1}} G(x+ty)\, dS(y) \right. \\
&\left. + \left(\frac{1}{t}\frac{\partial}{\partial t} \right)^{(n-2)/2} t^{n-1} \int_{\partial B^{n+1}} H(x+ty)\, dS(y) \right\}
\end{aligned}
$$

by Theorem 3.13. Since G and H do not depend on the last coordinate, we have by (3.7)

$$
\int_{\partial B^{n+1}} G(x+ty)\, dS(y) \;=\; 2\int_{B^n} G(x+ty) \frac{dy}{\sqrt{1-|y|^2}}
$$

and similarly for H. By the same argument as in the deterministic case the limit $t \to 0$ yields the initial value. □

We turn now to the *inhomogeneous* case. As in the last section we first determine the solution of the initial value problem for the inhomogeneous wave equation

$$
\begin{aligned}
\frac{\partial^2 U}{\partial t^2}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) \;&=\; F(t,x) \in C^{n/2+1}(\mathbb{R}_+ \times \mathbb{R}^n, (\mathcal{S})_{-1}) \\
U(0,x) \;&=\; 0 \\
\frac{\partial U}{\partial t}(0,x) \;&=\; 0
\end{aligned}
\tag{3.15}
$$

The Hermite transform converts (3.15) to the system

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) &= f(t,x) \\
u(0,x) &= 0 \\
\frac{\partial u}{\partial t}(0,x) &= 0.
\end{aligned}
$$

A solution of this problem is given by the Duhamel's Principle as

$$
u(t,x) = \frac{1}{\omega_n} \int_0^t \left(\frac{1}{r} \left(\frac{\partial}{\partial t} \right)^{(n-2)/2} r^{n-1} \int_{\partial B^n} f(t-r, x+ry)\, dS(y) \right) dr. \quad (3.16)
$$

See e.g. [ES], [Fo] or [J].

Theorem 3.17. *The initial value problem (3.15) can be uniquely solved in* $C^2(\mathbb{R}_+ \times \mathbb{R}^n, (\mathcal{S})_{-1})$, *by the process*

$$
U(t,x) = \frac{1}{\omega_n} \int_0^t \left(\frac{1}{r} \left(\frac{\partial}{\partial t} \right)^{(n-2)/2} r^{n-1} \int_{\partial B^n} F(t-r, x+ry)\, dS(y) \right) dr.
$$

Again, if we indicate by U_h and U_p the corresponding solutions in Theorem 3.16 and Theorem 3.17, one checks that $U = U_h + U_p$ gives a unique solution of the general initial value problem

$$
\begin{aligned}
\frac{\partial^2 U}{\partial t^2}(t,x) - \Delta U(t,x) &= F(t,x) \in C^{n/2+1}(\mathbb{R}_+ \times \mathrm{R}^{\mathrm{n}}, (\mathcal{S})_{-1}) \\
U(0,x) &= G(x) \in C^{(n+4)/2}(\mathrm{R}^{\mathrm{n}}, (\mathcal{S})_{-1}), \\
\frac{\partial U}{\partial t}(0,x) &= H(x) \in C^{(n+2)/2}(\mathrm{R}^{\mathrm{n}}, (\mathcal{S})_{-1}).
\end{aligned} \quad (3.17)
$$

So we obtain

Theorem 3.18. *System (3.17) allows a unique* $C^2(\mathbb{R}_+ \times \mathrm{R}^{\mathrm{n}}, (\mathcal{S})_{-1})$*-solution of the form*

$$
\begin{aligned}
U(t,x) \quad = \quad & C_n \left\{ \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2} t^{n-1} \int_{B^n} \frac{G(x+ty)}{\sqrt{1-|y|^2}}\, dy \right. \\
& \left. + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(n-2)/2} t^{n-1} \int_{B^n} \frac{H(x+ty)}{\sqrt{1-|y|^2}}\, dy \right\} \\
& + \frac{1}{\omega_n} \int_0^t \left(\frac{1}{r} \left(\frac{\partial}{\partial t} \right)^{(n-2)/2} r^{n-1} \int_{\partial B^n} F(t-r, x+ry)\, dS(y) \right) dr.
\end{aligned}
$$

Remark As an alternative approach to solve the stochastic wave equation (1.1) we shall mention that one could use the $\mathcal{S}-$ transform instead of the Hermite transform (see Remark 3.1.3 in [LØP]). The $\mathcal{S}-$transform is defined on a certain distribution space similar to $(\mathcal{S})_{-\rho}$ and it has the form

$$
\mathcal{S}(F)(\phi) = \langle\langle F(\omega), \widetilde{e}(\phi, \omega) \rangle\rangle
$$

for distributions F and for ϕ in a neighbourhood of zero in $\widetilde{\mathcal{S}}(X)$. The function $\widetilde{e}(\phi, \omega)$ is as in (2.4) and $\langle\langle \cdot, \cdot \rangle\rangle$ denotes an extension of the inner product on $L^2(\mu)$. By argueing similarly to the preceding proofs, one can attain analogous results with the help of $\mathcal{S}$.

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References

[A] Ahmed, N.U.:Stability of a class of stochastic distributed parameter systems with random boundary conditions. J. Math. Anal. Appl., Vol. 92, pp. 274-298, 1983.

[AB] Ahmed, N.U. Biswas S.K.: Existence of solutions and stability of a class of parabolic systems perturbed by generalized white noise on the boundary. Stoch. Anal. Appl., Vol. 2, pp. 347-368, 1984.

[AKS] Albeverio, S., Kondratiev, Y.G., Streit, L., How to generalize White Noise Analysis to Non-Gaussian Spaces. In: Dynamics of Complex and Irregular Systems. Eds.: Ph. Blanchard et al., World Scientific (1993).

[AW] Applebaum, D., Wu, J.-L., Stochastic partial differential equations driven by Lévy space-time white noise. Preprint University of Bochum, Germany, (1998).

[BH] Blomker, D. Hairer, M.: Stationary solutions for a model of amorphous thin-film growth. Preprint.

[B] Bertoin, J.: Lévy Processes. Cambridge University Press, Cambridge 1996.

[DH] Dalang, R. C., Hou, Q.: On Markov properties of Lévy waves in two dimensions. Stochastic Processes and their Applications, Vol. 72, pp. 265-287, 1997.

[DM] Dobrushin, R. L., Minlos, R. L., Polynomials in linear random functions. Russian Math. Surveys 32 (2), 71-127, (1977).

[DØP] Di Nunno, G., Øksendal, B., Proske, F., White Noise Analysis for Lévy processes, J. Funct. Anal., 206, 109-148, 2004.

[ES] Egorov, Y.V., Shubin, M.A.: Partial Differential Equations I. Encyclopedia of Mathematical Sciences, Vol 30, Springer-Verlag, 1992.

[Fo] Folland, G.B: Introduction To Partial Differential Equations. Princeton University Press, 1976.

[GV] Gelfand, I. M., Vilenkin, N. J., Generalized Functions. Vol. 4, Academic Press, 1968.

[HKPS] Hida, T., Kuo, H.-H, Potthoff, J., Streit, L.: White Noise. Kluwer, Dordrecht, 1993.

[HØUZ] Holden, H., Øksendal, B., Ubøe, J., Zhang T.-S.: Stochastic Partial Differential Equations- A Modeling, White Noise Functional Approach. Birkhäuser, Boston 1996.

[Ja] Jakobsen, E. R.: The Stochastic Wave Equation. Diploma Thesis, University of Trondheim, Norway, 1996.

[J] John, F.: Partial Differential Equations, Springer-Verlag, 4th ed., 2nd printing, 1986.

[K] Kondratiev, Y., Generalized functions in problems of infinitedimensional analysis. Ph.D. thesis, Kiev University, (1978).

[KDS] Kondratiev, Y., Da Silva, J. L., Streit, L.: Generalized Appell systems, Methods Funct. Anal. Topology, 3 (3), (1997), 28-61.

[KLS] Kondratiev, Y., Leukert, P., Streit, L., Wick calculus in Gaussian analysis, Acta Appl. Math., 44 (1996), no. 3, 269-294.

[Ku] Kuo, H. H., White Noise Distribution Theory. Prob. and Soch. Series, Boca Raton, FL: CRC Press, (1996).

[LØP] Løkka, A., Øksendal, B., Proske, F.: Stochastic partial differential equations driven by Lévy space time white noise. Annals of Applied Probability, Vol. 14, No. 3, 2004.

[LØU] Lindstrøm, T., Øksendal, B., Ubøe, J., Stochastic differential equations involving positive noise. In Barlow, M. and Bingham, N. (editors): Stochastic Analysis. Cambridge Univ. Press, pp. 261-303, (1991).

[LP] Løkka, A., Proske, F., Infinite dimensional analysis of pure jump Lévy processes on the Poisson space. To appear in Mathematical Scandinavia.

[M] Mueller, C., The heat equation with Lévy noise. Stoch. Proc. Appl. 74, 67-82, (1998).

[NS] Nualart, D., Schoutens, W.: Chaotic and predictable representations for Lévy processes. Stochastic Process. Appl., 90, (2000), 109-122.

[O] Obata, N., White Noise Calculus and Fock Space, LNM, 1577, Berlin: Springer-Verlag, (1994).

[ØP] Øksendal, B., Proske, F., White noise of Poisson random measures. Potential Analysis, Vol. 21, No 4, 375-403, 2004.

[Sa] Sato, K.: Lévy Processes and Infinitely Divisible Distributions, Cambridge University Studies in Advanced Mathematics, Vol. 68, Cambridge University Press, Cambridge 1999.

[S] Sornette, D.: "String" formulation of the dynamics of the forward interest curve. Eur. Phys. J. B, Vol. 3, pp. 125-137, 1998.

[T] Thangavelu, S.: Lectures on Hermite and Laguerre Expansions. Mathematical Notes 42, Princeton University Press 1993.

[W] Walsh, J. B., An introduction to stochastic partial differential equations. In: Ecole dÉté de Probabilités de St. Flour XIV, pp. 266-439, Lect. Notes in Math. 1180, Springer-Verlag, Berlin, (1986).